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# A Class of Compactly Supported Symmetric Biorthogonal Wavelets*

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**Abstract:** This paper considers the construction of 3-band compactly supported symmetric biorthogonal wavelets. Given an interpolatory refinable function, we present an explicit iterative algorithm for constructing its dual scaling function with desired regularity, and discuss the properties of the constructed scaling function, such as its symmetry and decay. Finally, several design examples are given.

**Keywords:** compactly supported; biorthogonal; symmetric; interpolating; scaling function

**Classification:** AMS(2000) 42C15; 94A12 **CLC number:** O172.42 **Document code:** A

## 1 Introduction

In [1], Daubechies constructed a family of compactly supported orthonormal scaling functions and their corresponding orthonormal wavelets with dilation factor 2. Daubechies has proved that there exists no dyadic compactly supported orthogonal scaling function which is symmetric, except for the Haar scaling function. For the 3-band case, it is known from [2,3] that there also exists no compactly supported symmetric orthogonal scaling function which is interpolatory. Hence, such scaling functions have to be biorthogonal. In [4], Ji and Shen constructed a class of compactly supported biorthogonal wavelets and orthogonal interpolatory wavelets with dilation factor 2. Methods for constructing $M$-band biorthogonal wavelets were also given in [5,6]. Examples of biorthogonal B-spline type wavelets with certain regularities were obtained in [7]. In [8], the authors considered the design of 3-channal adaptive biorthogonal filter banks via lifting. In this paper, we will consider the construction of 3-band compactly supported symmetric biorthogonal wavelets, i.e., given a interpolatory refinable function, we present an explicit iterative algorithm for constructing its dual scaling function with desired regularity.

Recall that a multiresolution analysis (MRA) with dilation factor $M$, is a sequence of nested subspaces of $L^2(R)$, i.e., $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$, where

$$V_j = \mathrm{span}\{f(M^j x - k) : k \in Z\}$$

for some $f(x) \in L^2(R)$, and

$$\overline{\bigcup_{j\in Z} V_j} = L^2(R), \quad \bigcap_{j\in Z} V_j = \{0\}.$$

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The function $f(x)$ is called the scaling function. Let $f(x)$ be a compactly supported scaling function with dilation factor $M$. Then $\int_R f(x) \neq 0$, and $f(x)$ satisfies the refinement equation

$$f(x) = \sum_{k\in Z} c_k f(Mx-k), \quad \sum_{k\in Z} c_k = M, \tag{1}$$

where $c_k$ are real, and $c_k \neq 0$ for only finitely many $k \in Z$. Define a trigonometric polynomial

$$P(z) = \frac{1}{M}\sum_{k\in Z} c_k z^k, \quad z = \mathrm{e}^{-iw}.$$

## 2 Preliminary results

Define masks as

$$m_0(w) = \frac{1}{M}\sum_n h_n \mathrm{e}^{-inw}, \quad \tilde{m}_0(w) = \frac{1}{M}\sum_n g_n \mathrm{e}^{-inw}. \tag{2}$$

Here, we assume that only finitely many $h_n$, $g_n$ are nonzero. Some of our results can be extended to infinite sequences that have sufficient decay for $|n| \to \infty$. Then we define $\phi$, $\tilde{\phi}$ by

$$\hat{\phi}(w) = \prod_{j=1}^{\infty} m_0(M^{-j}w), \quad \hat{\tilde{\phi}}(w) = \prod_{j=1}^{\infty} \tilde{m}_0(M^{-j}w). \tag{3}$$

These infinite products can converge only if $m_0(0) = \tilde{m}_0(0) = 1$, i.e., if

$$\sum_n h_n = \sum_n g_n = M. \tag{4}$$

If (4) is satisfied, then the infinite products in (3) converge uniformly and absolutely on compact sets. So $\hat{\phi}(w)$ and $\hat{\tilde{\phi}}(w)$ are well defined $C^\infty$ functions. What is more important, if the infinite products in (3) converge in $L^2(R)$, then two scaling functions $\phi(x)$, $\tilde{\phi}(x)$ will be well defined according to [9]. From Lemma 3.1 in [10], $\phi(x)$ and $\tilde{\phi}(x)$ have compact support.

Let $\phi(x)$, $\tilde{\phi}(x) \in L^2(R)$ be the scaling functions, and $m_0(w)$, $\tilde{m}_0(w)$ be their masks, respectively. $\tilde{\phi}(x)$ is called the dual function of $\phi(x)$ if

$$\langle \phi(x), \tilde{\phi}(x-k)\rangle = \delta_{k,0}, \quad k \in Z.$$

If $\phi(x)$ and $\tilde{\phi}(x)$ are biorthogonal scaling functions, then their masks must satisfy

$$\sum_{k=0}^{M-1} m_0\Big(w + \frac{2\pi k}{M}\Big)\overline{\tilde{m}_0\Big(w + \frac{2\pi k}{M}\Big)} = 1. \tag{5}$$

**Definition 2.1** A continuous scaling function $\phi(x)$ is interpolatory if it satisfies $\phi(k) = \delta_{k,0}$, $k \in Z$.

From [4], it is easy to know that a compactly supported continuous function $\phi(x)$ is interpolatory if and only if

$$\sum_{k\in Z} \hat{\phi}(w + 2k\pi) = 1.$$

Let $m_0(w)$ be its mask, then it satisfies

$$\sum_{k=0}^{M-1} m_0\Big(w + \frac{2\pi k}{M}\Big) = 1, \quad \forall\, w \in R. \tag{6}$$

Condition (6) is called the interpolatory condition. Condition (6) is only a necessary condition. A necessary and sufficient condition in terms of the refinement mask is given in [11]. From [4], the autocorrelation of an orthonormal scaling function is interpolatory.

## 3 Dual functions of interpolatory scaling functions

In this section we only consider the case of $M = 3$. Here we consider

$$P(w) = z^{-N}\Big(\frac{1+z+z^2}{3}\Big)^N Q(z), \tag{7}$$

where

$$Q(z) = \sum_{j=-k}^{k} a_j z^j, \quad a_{-k} = a_k$$

and all $a_j$ are real.

If the mask $P(w)$ is of the form in (7), then it is said to have $N$-regularity. It is easy to know that the Fourier transform of $\phi(x)$ is given by

$$\hat{\phi}(w) = \prod_{j=1}^{\infty} P(3^{-j} w). \tag{8}$$

**Theorem 3.1** Let $P(w)$ be a $2\pi$-periodic function of the form in (7) that satisfies the interpolatory condition (6). Define

$$H(w) := P^2(w) + 3P(w)(1 - P(w)) + 2P\Big(w + \frac{2\pi}{3}\Big)P\Big(w + \frac{4\pi}{3}\Big). \tag{9}$$

Then the function $P(w)H(w)$ satisfies the interpolatory condition (6) and the coefficients of $H(w)$ are all real. Also $P(w)H(w)$ is of the form in (7).

**Proof** First let $H(w)$ be the function defined by (9). Then

$$\begin{aligned}
P(w)H(w) &= 3P^2(w) - 2P^3(w) + 2P(w)P\Big(w + \frac{2\pi}{3}\Big)P\Big(w + \frac{4\pi}{3}\Big),\\
P\Big(w + \frac{2\pi}{3}\Big)H\Big(w + \frac{2\pi}{3}\Big) &= 3P^2\Big(w + \frac{2\pi}{3}\Big) - 2P^3\Big(w + \frac{2\pi}{3}\Big)\\
&\quad +2P(w)P\Big(w + \frac{2\pi}{3}\Big)P\Big(w + \frac{4\pi}{3}\Big),\\
P\Big(w + \frac{4\pi}{3}\Big)H\Big(w + \frac{4\pi}{3}\Big) &= 3P^2\Big(w + \frac{4\pi}{3}\Big) - 2P^3\Big(w + \frac{4\pi}{3}\Big)\\
&\quad +2P(w)P\Big(w + \frac{2\pi}{3}\Big)P\Big(w + \frac{4\pi}{3}\Big).
\end{aligned}$$

Therefore,

$$\sum_{k=0}^{2} P\Big(w+\frac{2k\pi}{3}\Big)H\Big(w+\frac{2k\pi}{3}\Big) = \Big[P(w)+P\Big(w+\frac{2\pi}{3}\Big)+P\Big(w+\frac{4\pi}{3}\Big)\Big]^3 = 1.$$

Next, from (7) and (9), to prove the coefficients of $H(w)$ are real, it suffices to show the coefficients of

$$Q\Big(w+\frac{2\pi}{3}\Big)Q\Big(w+\frac{4\pi}{3}\Big)$$

are real. Observe that

$$Q\Big(w+\frac{2\pi}{3}\Big)Q\Big(w+\frac{4\pi}{3}\Big) = \sum_{j=-k}^{k}\sum_{n=-k}^{k} a_j a_n e^{-i(j+n)w-i(\frac{2j\pi}{3}+\frac{4n\pi}{3})}. \tag{10}$$

Since

$$e^{-i(\frac{2j\pi}{3}+\frac{4n\pi}{3})}+e^{-i(\frac{2n\pi}{3}+\frac{4j\pi}{3})} = 2 \quad \text{or} \quad -1, \quad \forall\, j, n \in Z.$$

Hence the coefficients of (10) are real.

Since $P(w)$ is symmetric about the origin, consequently, $P(w)H(w)$ is of the form in (7). So Theorem 3.1 holds.

Let $\phi(x)$ be the symmetric interpolatory scaling function whose mask $P(w)$ is defined by (7). We can use Theorem 3.1 to construct the dual function of $\phi(x)$. First define the dual mask by

$$\tilde{P}(w) := P^2(w) + 3P(w)(1-P(w)) + 2P\Big(w+\frac{2\pi}{3}\Big)P\Big(w+\frac{4\pi}{3}\Big).$$

**Lemma 3.2** Suppose that the function $\tilde{\phi}(x)$ defined by

$$\hat{\tilde{\phi}}(w) = \prod_{j=1}^{\infty}\tilde{P}(w/3^j)$$

is in $L^2(R)$. Then $\tilde{\phi}(x)$ is a dual function of $\phi(x)$.

**Proof** The proof is similar to the Theorem 2.2 in [4].

**Theorem 3.3** Let the mask $P(w)$ be of the form in (7), and $Q(z)$ satisfy

$$Q(v_k)\neq 0, \quad Q(0)=1, \quad v_k = \frac{2k\pi}{3}, \quad k=1,2.$$

If there exists some $n \geq 0$ such that

$$B_n = \max_w |Q(w)Q(w/3)\cdots Q(w/3^n)| < 3^{N-1/2}, \tag{11}$$

then $\hat{\phi}(w)$ satisfies the decay condition

$$|\hat{\phi}(w)| \leq C(1+|w|)^{-1/2-\varepsilon}, \quad \text{with} \quad \varepsilon = N-\frac{1}{2}-\frac{\log B_n}{\log 3} > 0, \tag{12}$$

where $C>0$ is a constant. What is more, $\phi(x)\in L^2(R)$ and satisfies the refinement equation.

**Proof** From the Appendix B in [6], it is easy to know that if the condition (10) can be satisfied, then (11) holds consequently. So $\hat{\phi}(x)$ belongs to $L^2(R)$. By applying the Theorem 2.17 in [12], there exists a function $\phi \in L^2(R)$ and it satisfies the following refinement equation

$$P\Big(\frac{w}{3}\Big)\hat{\phi}\Big(\frac{w}{3}\Big) = P\Big(\frac{w}{3}\Big)\prod_{j=1}^{\infty} P(w/3^{j+1}) = \hat{\phi}(w).$$

**Corollary 3.4** Let $P(w)$ be a mask of the form in (7) and $a_j$, $j = 0, 1, \cdots, k$ be all nonnegative. If the coefficients $a_j$ satisfy

$$\sum_{j=0}^{k} a_j < 3^{N-1/2},$$

then there exists a scaling function $\phi(x) \in L^2(R)$ corresponding to $P(w)$ defined by

$$\hat{\phi}(w) = \prod_{j=1}^{\infty} P(w/3^j).$$

Now, we give an iterative method.

**Iterative construction** Let $\phi(x)$ be the given interpolatory refinable function whose mask $P(w)$ is given by (7). Let $P_0 = P(w)$. For $k = 1, 2, \cdots$, then

(i) define

$$P_k := P_{k-1}\Big[P_{k-1}^2(w) + 3P_{k-1}(w)(1 - P_{k-1}(w)) + 2P_{k-1}\Big(w + \frac{2\pi}{3}\Big)P_{k-1}\Big(w + \frac{4\pi}{3}\Big)\Big];$$

(ii) define

$$\tilde{P}_k := \frac{P_k}{P_0};$$

(iii) define

$$\hat{\tilde{\phi}}(w) = \prod_{j=1}^{\infty} \tilde{P}_k(w/3^j).$$

From the above theorem and algorithm, if in the $k$-th step, the conditions in Theorem 3.3 are satisfied, then we can iterate these steps to get a function $\hat{\phi}(w)$ with better regularity.

## 4 Examples

In what following, we give two examples of biorthogonal 3-band refinable functions.

Let $P_0$ be the mask of the scaling function $\phi(x)$, which is symmetric and interpolatory, where

$$P_0(w) = z^{-2}\Big(\frac{1 + z + z^2}{3}\Big)^2.$$

First, let $k = 1$. By using the iterative construction, we get

$$\tilde{P}_1 = z^{-2}\Big(\frac{1 + z + z^2}{3}\Big)^2\Big(\frac{-4z^{-1} + 11 - 4z}{3}\Big).$$

It is easy to verify that $\tilde{P}_1$ satisfies the conditions in Theorem 3.3. So it can generate a dual function $\tilde{\phi}_1(x)$ of $\phi(x)$. Clearly, $P_0\tilde{P}_1$ is the mask of another interpolatory scaling function $\varphi(x)$. Similarly, we can get a dual function $\tilde{\varphi}(x)$ of $\varphi(x)$.

Let $k = 2$. By using the above algorithm, we get $\tilde{P}_2$, which also satisfies the condition in Theorem 3.3. So it can generate a dual function $\tilde{\phi}_2(x)$ of $\phi(x)$. Let

$$\tilde{P}_2 = \frac{1}{3}\sum_n g_n z^n.$$

We obtain that

$$\begin{aligned}[g_n]_{n=-17}^{17} = [&-2.9725\text{e-}006,\ -5.9446\text{e-}006,\ -8.9153\text{e-}006,\ 5.5141\text{e-}005,\ 0.00035997,\\ &0.00060456,\ -0.00069907,\ -0.0065288,\ -0.013667,\ 0.015167,\ 0.053461,\\ &0.079964,\ -0.097909,\ -0.25377,\ -0.28033,\ 0.32117,\ 0.9687,\ 1.4269,\\ &0.9687,\ 0.32117,\ -0.28033,\ -0.25377,\ -0.097909,\ 0.079964,\ 0.053461,\\ &0.015167,\ -0.013667,\ -0.0065288,\ -0.00069907,\ 0.00060456,\\ &0.00035997,\ 5.5141\text{e-}005,\ -8.9153\text{e-}006,\ -5.9446\text{e-}006,\ -2.9725\text{e-}006].\end{aligned}$$

The following figures show the graphs of $\phi(x)$ and the dual scaling functions of $\phi(x)$. From the graphs we know that the larger of the $k$, the smoother of the dual function.

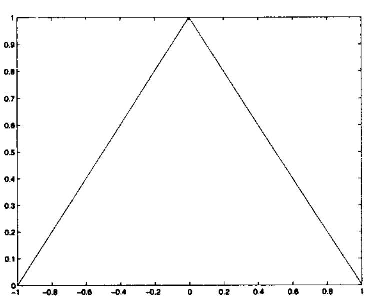

*Figure 1: Graph of $\phi(x)$*

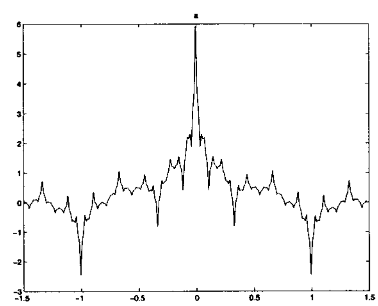

*Figure 2: Graph of $\tilde{\phi}_1(x)$*

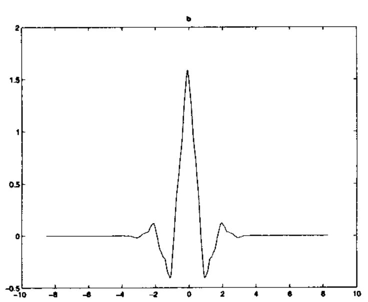

*Figure 3: Graph of $\tilde{\phi}_2(x)$*

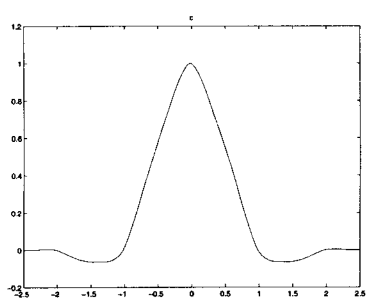

*Figure 4: Graph of $\varphi(x)$*

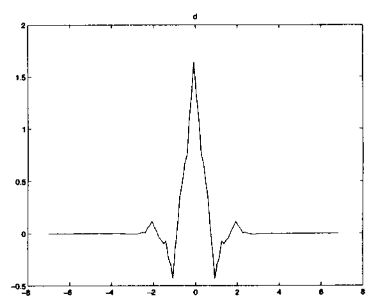

*Figure 5: Graph of $\tilde{\varphi}(x)$*

## References:


[1] Daubechies I. Orthonormal bases of compactly supported wavelets[J]. Communications on Pure and Applied Mathematics, 1988, 41(7): 909-996

[2] Lewis R M. Cardinal interpolation multiresolution[J]. Journal of Approximation Theory, 1994, 76(2): 177-202

[3] Shui P L, *et al.* $M$-band compactly supported orthogonal symmetric interpolating scaling functions[J]. IEEE Transactions on Signal Processing, 2001, 49(8): 1704-1713

[4] Ji H, Shen Z W. Compactly supported (bi)orthogonal wavelets generated by interpolatory refinable functions[J]. Advances in Computational Mathematics, 1999, 11(1): 81-104

[5] Soardi P M. Biorthogonal $M$-channal compactly supported wavelets[J]. Constructive Approximation, 2000, 16(2): 283-311

[6] Chan S C, *et al.* $M$-channal compactly supported biorthogonal cosine-modulated wavelet bases[J]. IEEE Transactions on Signal Processing, 1998, 46(4): 1142-1151

[7] He T X. Construction of biorthogonal B-spline type wavelet sequences with certain regularities[J]. Journal of Applied Functional Analysis, 2007, 2(4): 339-360

[8] Xu X P, *et al.* Design of three-channel adaptive biorthogonal filterbanks via lifting[J]. Chinese Journal of Engineering Mathematics, 2005, 22(1): 35-39

[9] Daubechies I, Lagarias J C. Two-scale difference equation I: existence and global regularity of solutions[J]. SIAM Journal on Mathematical Analysis, 1991, 22(5): 1388-1410

[10] Cohen A, *et al.* Biorthogonal bases of compactly supported wavelets[J]. Communications on Pure and Applied Mathematics, 1992, 45(5): 485-560

[11] Lawton W, *et al.* Stability and orthonormality of multivariate refinable functions[J]. SIAM Journal on Mathematical Analysis, 1997, 28(4): 999-1014

[12] Chui C K. A Introduction to Wavelet[M]. Boston: Academic Press, 1992


# 一类紧支撑对称双正交小波


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**摘　要**: 研究伸缩因子为 3 紧支撑对称双正交小波的构造问题。任给一个插值对称的加细函数，本文提供一个构造其对偶尺度函数的迭代算法，其对偶尺度函数的正则性可以达到任意水平，讨论了它的一些性质，如对称性和衰减性。最后，给出构造算例。



**关键词**: 紧支撑；双正交的；对称的；插值；尺度函数